13.1

In Exercises 13–16, find and sketch the level curves $f(x, y) = c$ on the same set of coordinate axes for the given values of c. We refer to these level curves as a contour map.

13. $f(x, y) = x + y - 1, \quad c = -3, -2, -1, 0, 1, 2, 3$

14. $f(x, y) = x^2 + y^2, \quad c = 0, 1, 4, 9, 16, 25$

15. $f(x, y) = xy, \quad c = -9, -4, -1, 0, 1, 4, 9$

16. $f(x, y) = \sqrt{25 - x^2 - y^2}, \quad c = 0, 1, 2, 3, 4$

Sol'n: $f(x,y) = \sqrt{25 - x^2 - y^2} = c.$

$$\Rightarrow x^2 + y^2 = 25 - c^2$$

Circles of radius $\sqrt{25 - c^2}$.

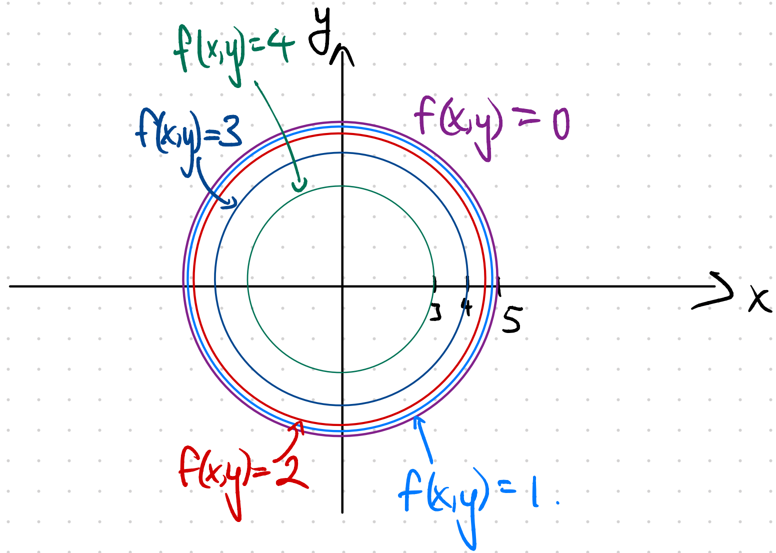
$c = 0$: circle of radius 5

$c = 1$: circle of radius $\sqrt{24}$

$c = 2$: circle of radius $\sqrt{21}$

$c = 3$: circle of radius 4

$c = 4$: circle of radius 3.



13.1 In Exercises 17–30, **(a)** find the function's domain, **(b)** find the function's range, **(c)** describe the function's level curves, **(d)** find the boundary of the function's domain, **(e)** determine whether the domain is an open region, a closed region, or neither, and **(f)** decide whether the domain is bounded or unbounded.

24. $f(x, y) = \sqrt{9 - x^2 - y^2}$

e) Domain is closed.

f) Domain is bounded.

Sol'n: a) Domain: Need $9 - x^2 - y^2 \geq 0$

$\Rightarrow 0 \leq x^2 + y^2 \leq 9$

So $D(f) = \{(x,y) \in \mathbb{R}^2 : 0 \leq x^2 + y^2 \leq 9\}$

the closed disk of radius 3 centred at origin.

b) Range: $R(f) = [0,3]$.

c) Let $c \in [0,3]$. Then the level curves $f(x,y) = c$ are circles of radius $\sqrt{9-c^2}$ centred at the origin.

d) $\partial D(f) = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 9\}$ the circle of radius 3 centred at origin.

13.1

In Exercises 17–30, **(a)** find the function's domain, **(b)** find the function's range, **(c)** describe the function's level curves, **(d)** find the boundary of the function's domain, **(e)** determine whether the domain is an open region, a closed region, or neither, and **(f)** decide whether the domain is bounded or unbounded.

28. $f(x, y) = \tan^{-1}\left(\frac{y}{x}\right)$

a) Domain: $x \neq 0$. $D(f) = \{(x,y) \in \mathbb{R}^2 : x \neq 0\}$

b) Range: $R(f) = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

c) Let $c \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then $f(x,y) = c$ are straight lines passing through the origin, excluding the origin ($x = 0 \Rightarrow y = 0$) s.t. the slope of the line is given by c.

d) $\partial D(f) = \{0\}$.

e) The domain is open

f) The domain is unbounded.

13.1

Functions of Two Variables

Display the values of the functions in Exercises 37–48 in two ways: **(a)** by sketching the surface $z = f(x, y)$ and **(b)** by drawing an assortment of level curves in the function's domain. Label each level curve with its function value.

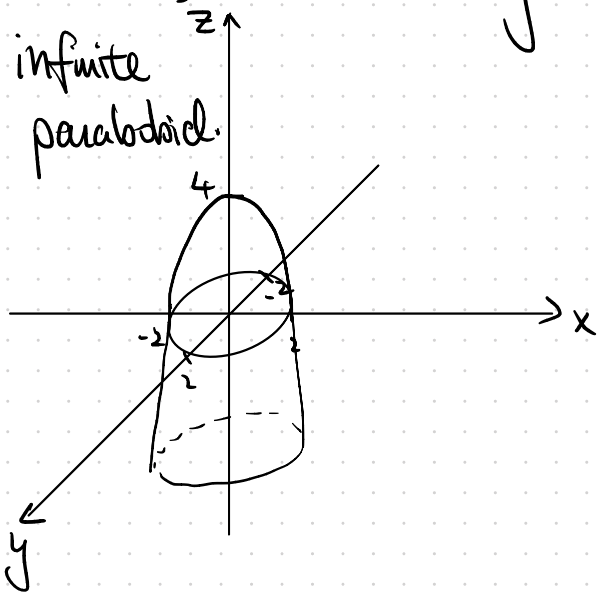
37. $f(x, y) = y^2$ **38.** $f(x, y) = \sqrt{x}$

39. $f(x, y) = x^2 + y^2$ **40.** $f(x, y) = \sqrt{x^2 + y^2}$

41. $f(x, y) = x^2 - y$ **42.** $f(x, y) = 4 - x^2 - y^2$

Sol'n: a) $z = 4 - x^2 - y^2$

infinite paraboloid.



b) $f(x,y) = c$, $c = 0, -1, 1, -2, 2$

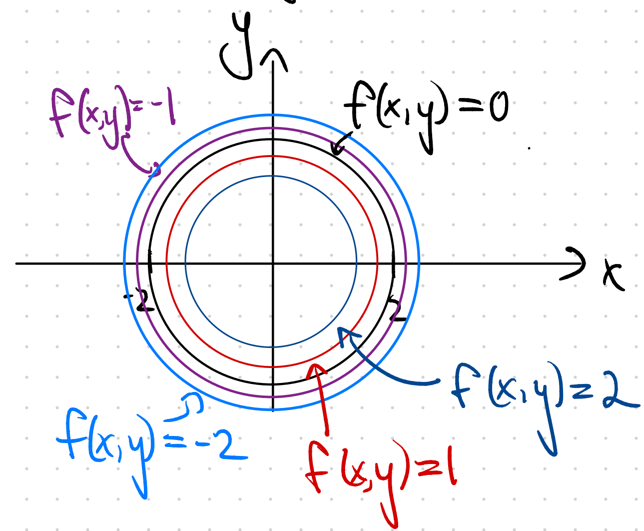
$c=0$: $4 = x^2+y^2$ circle of radius 2

$c=-1$: $5 = x^2+y^2$ circle of radius $\sqrt{5}$

$c=1$: $3 = x^2+y^2$ circle of radius $\sqrt{3}$

$c=-2$: $6 = x^2+y^2$ circle of radii $\sqrt{6}$

$c=2$: $2 = x^2+y^2$ circle of radius $\sqrt{2}$



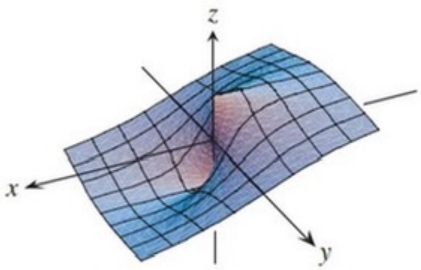
13.2

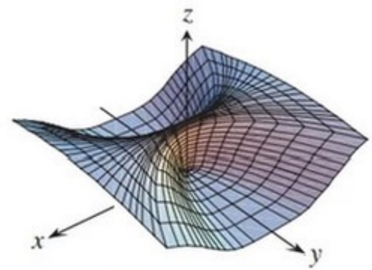
No Limit Exists at the Origin

By considering different paths of approach, show that the functions in Exercises 41–48 have no limit as $(x, y) \to (0, 0)$.

41. $f(x, y) = -\dfrac{x}{\sqrt{x^2 + y^2}}$

42. $f(x, y) = \dfrac{x^4}{x^4 + y^2}$





Sol'n: Let $m \in \mathbb{R}$ and consider the line $y = mx$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=mx}} f(x,y) = \lim_{\substack{(x,y)\to(0,0)\\ y=mx}} \frac{x^4}{x^4+y^4} = \lim_{x\to 0} \frac{x^4}{x^4+m^4x^4} = \lim_{x\to 0} \frac{\cancel{x^4}}{\cancel{x^4}(1+m^4)}$$

$$= \lim_{x\to 0} \frac{1}{1+m^4} = \frac{1}{1+m^4}.$$

So in particular, along the line $y=0$, limit is 1, while along the line $y=x$, limit is $\frac{1}{2}$, so limit DNE. //

13.2 **Theory and Examples**

In Exercises 49–54, show that the limits do not exist.

49. $\lim_{(x,y)\to(1,1)} \frac{xy^2-1}{y-1}$ **50.** $\lim_{(x,y)\to(1,-1)} \frac{xy+1}{x^2-y^2}$

Sol'n: Line 1: $y=-1$: $\lim_{\substack{(x,y)\to(1,-1)\\ y=-1}} \frac{xy+1}{x^2-y^2} = \lim_{x\to 1} \frac{x(-1)+1}{x^2-(-1)^2} = \lim_{x\to 1} \frac{-x+1}{x^2-1}$

horizontal line

$$= \lim_{x\to 1} \frac{-1}{2x} = -\frac{1}{2}$$

Line 2: $x=1$: $\lim_{\substack{(x,y)\to(1,-1)\\ x=1}} \frac{xy+1}{x^2-y^2} = \lim_{y\to -1} \frac{1\cdot y+1}{1^2-y^2} = \lim_{y\to -1} \frac{y+1}{1-y^2}$

vertical line

$$= \lim_{y\to -1} \frac{1}{-2y} = \frac{1}{2}$$

So $\lim_{(x,y)\to(1,-1)} \frac{xy+1}{x^2-y^2}$ DNE

13.2 56. Let $f(x,y) = \begin{cases} x^2, & x \geq 0 \\ x^3, & x < 0 \end{cases}$.

Find the following limits.

a. $\lim_{(x,y)\to(3,-2)} f(x,y)$

b. $\lim_{(x,y)\to(-2,1)} f(x,y)$

c. $\lim_{(x,y)\to(0,0)} f(x,y)$

Note that since $f(x,y) = f(x)$ does not depend on y, the limits are independent of y.

Sol'n: a) Limiting point is $(3,-2)$, so as $x \to 3$, $x > 0$, so

$$\lim_{(x,y)\to(3,-2)} f(x) = \lim_{x\to 3} x^2 = \boxed{9}$$

b) Limiting point is $(-2,1)$, so as $x \to -2$, $x < 0$, so

$$\lim_{(x,y)\to(-2,1)} f(x) = \lim_{x\to -2} x^3 = \boxed{-8}$$

c) Consider both sides: $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2 = 0.$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x^3 = 0.$

$\Rightarrow$ So $\lim_{(x,y)\to(0,0)} f(x,y) = \boxed{0.}$

13.2

Using the Limit Definition

Each of Exercises 73–78 gives a function $f(x, y)$ and a positive number ε. In each exercise, show that there exists a $\delta > 0$ such that for all (x, y),

$$\sqrt{x^2 + y^2} < \delta \Rightarrow |f(x, y) - f(0,0)| < \varepsilon.$$

73. $f(x, y) = x^2 + y^2, \quad \varepsilon = 0.01$

74. $f(x, y) = y/(x^2 + 1), \quad \varepsilon = 0.05$

75. $f(x, y) = (x + y)/(x^2 + 1), \quad \varepsilon = 0.01$

76. $f(x, y) = (x + y)/(2 + \cos x), \quad \varepsilon = 0.02$

Sol'n: $f(0,0) = \dfrac{0+0}{2+\cos 0} = \dfrac{0}{2+1} = 0.$

So $|f(x,y) - f(0,0)| = \left| \dfrac{x+y}{2+\cos x} - 0 \right| = \left| \dfrac{x+y}{2+\cos x} \right| \leq |x+y| \leq |x| + |y| \leq \sqrt{x^2+y^2} + \sqrt{x^2+y^2}.$

Since $-1 \leq \cos x \leq 1$ for all $x \in \mathbb{R}$, we have

$1 \leq 2 + \cos x \leq 3$ for all $x \in \mathbb{R}$,

$\Rightarrow \frac{1}{3} \leq \dfrac{1}{2+\cos x} \leq 1$

So setting $\delta = \frac{\varepsilon}{2} = 0.01$, we have if $\sqrt{x^2+y^2} < \delta = 0.01$, then

$|f(x,y) - f(0,0)| \leq 2\sqrt{x^2+y^2} < 2 \cdot \delta = \varepsilon = 0.02$ //.

Can actually get a sharper estimate by using Cauchy-Schwarz inequality to obtain

$|x+y| = |1 \cdot x + 1 \cdot y| \leq \sqrt{2}\sqrt{x^2+y^2}$,

then can set $\delta = \frac{\varepsilon}{\sqrt{2}}$ instead of $\frac{\varepsilon}{2}$.

13.2 Each of Exercises 79–82 gives a function $f(x, y, z)$ and a positive number ε. In each exercise, show that there exists a $\delta > 0$ such that for all (x, y, z),

$$\sqrt{x^2 + y^2 + z^2} < \delta \Rightarrow |f(x, y, z) - f(0, 0, 0)| < \varepsilon.$$

79. $f(x, y, z) = x^2 + y^2 + z^2, \quad \varepsilon = 0.015$

80. $f(x, y, z) = xyz, \quad \varepsilon = 0.008$

81. $f(x, y, z) = \dfrac{x + y + z}{x^2 + y^2 + z^2 + 1}, \quad \varepsilon = 0.015$

82. $f(x, y, z) = \tan^2 x + \tan^2 y + \tan^2 z, \quad \varepsilon = 0.03$

Sol'n: $f(0,0,0) = 0$.

So $|f(x,y,z) - f(0,0,0)| = |xyz| \leq |x||y||z|$

We have $|x| = \sqrt{x^2} \leq \sqrt{x^2+y^2+z^2}$ and similarly for $|y|, |z|$.

So $|f(x,y,z) - f(0,0,0)| = |xyz| \leq |x||y||z|$

$$\leq \left(\sqrt{x^2+y^2+z^2}\right)^3.$$

So setting $\delta = \varepsilon^{1/3}$, we have that if $\sqrt{x^2+y^2+z^2} < \delta$, then

$|f(x,y,z) - f(0,0,0)| \leq \left(\sqrt{x^2+y^2+z^2}\right)^3 < \delta^3 = \varepsilon$ //.

13.2 Each of Exercises 79–82 gives a function $f(x, y, z)$ and a positive number ε. In each exercise, show that there exists a $\delta > 0$ such that for all (x, y, z),

$$\sqrt{x^2 + y^2 + z^2} < \delta \Rightarrow |f(x, y, z) - f(0,0,0)| < \varepsilon.$$

79. $f(x, y, z) = x^2 + y^2 + z^2, \quad \varepsilon = 0.015$

80. $f(x, y, z) = xyz, \quad \varepsilon = 0.008$

81. $f(x, y, z) = \dfrac{x + y + z}{x^2 + y^2 + z^2 + 1}, \quad \varepsilon = 0.015$

82. $f(x, y, z) = \tan^2 x + \tan^2 y + \tan^2 z, \quad \varepsilon = 0.03$

Sol'n: $f(0,0,0) = \tan^2(0) + \tan^2(0) + \tan^2(0) = 0.$

So $|f(x,y,z) - f(0,0,0)| = |\tan^2 x + \tan^2 y + \tan^2 z|$

$\leq |\tan^2 x| + |\tan^2 y| + |\tan^2 z|.$

Note for $x \in (0,1)$, $\tan^2 x$ is increasing, so

for x, y, z such that $\sqrt{x^2+y^2+z^2} < 1$, $x < \sqrt{x^2+y^2+z^2}$

$y < \sqrt{x^2+y^2+z^2}$

$z < \sqrt{x^2+y^2+z^2}$,

So

$|f(x,y,z) - f(0,0,0)| \leq |\tan^2 x| + |\tan^2 y| + |\tan^2 z|$

$\leq 3|\tan^2(\sqrt{x^2+y^2+z^2})|.$

So taking $\delta < \arctan\sqrt{\frac{\varepsilon}{3}} = \arctan\left(\sqrt{\frac{0.03}{3}}\right) < 1$

we have that if $\sqrt{x^2+y^2+z^2} < \delta$,

$|f(x,y,z) - f(0,0,0)| \leq 3\left|\tan^2\left(\arctan\left(\sqrt{\frac{\varepsilon}{3}}\right)\right)\right|$

$< 3\left(\sqrt{\frac{\varepsilon}{3}}\right)^2 = \varepsilon = 0.03$

//.